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Stability of fluid flow past a membrane

By R. M. THAOKAR AND V. KUMARAN

Department of Chemical Engineering, Indian Institute of Science, Bangalore 560 012, India

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The stability of fluid flow past a membrane of infinitesimal thickness is analysed in the limit of zero Reynolds number using linear and weakly nonlinear analyses. The system consists of two Newtonian fluids of thickness R^* and HR^*, separated by an infinitesimally thick membrane, which is flat in the unperturbed state. The dynamics of the membrane is described by its normal displacement from the flat state, as well as a surface displacement field which provides the displacement of material points from their steady-state positions due to the tangential stress exerted by the fluid flow. The surface stress in the membrane (force per unit length) contains an elastic component proportional to the strain along the surface of the membrane, and a viscous component proportional to the strain rate. The linear analysis reveals that the fluctuations become unstable in the long-wave ($\alpha \to 0$) limit when the non-dimensional strain rate in the fluid exceeds a critical value Λ_t, and this critical value increases proportional to α^2 in this limit. Here, α is the dimensionless wavenumber of the perturbations scaled by the inverse of the fluid thickness R^{*-1}, and the dimensionless strain rate is given by $\Lambda_t = (\dot{\gamma}^* R^* \eta^* / \Gamma^*)$, where η^* is the fluid viscosity, Γ^* is the tension of the membrane and $\dot{\gamma}^*$ is the strain rate in the fluid. The weakly nonlinear stability analysis shows that perturbations are supercritically stable in the $\alpha \to 0$ limit.

1. Introduction

Fluid flow past flexible surfaces is encountered in industrial applications like membrane reactors and hollow fibre reactors, as well as in separation processes in pharmaceutical industries, where there is flow and diffusion across tubes and channels made up of polymer matrices and membranes. Flow past membranes is also encountered in biological systems, where surfaces and organelles of cells are made up of deformable lipid membranes. In this analysis, the effect of membrane deformation on the stability of the flow of a fluid adjacent to a membrane is examined using linear and nonlinear approaches. The membrane is described by a constitutive equation where the normal stress difference across the membrane surface is balanced by membrane tension, while the tangential stress exerted by the fluid is balanced by a surface stress along the membrane surface due to the strain in the plane of the membrane. This type of constitutive relation has been used previously to describe the deformation of viscoelastic polymer films (Harden & Pleiner 1994).

Kumaran (1995) and Kumaran, Fredrickson & Pincus (1994) studied the flow of a fluid past a viscoelastic material at zero Reynolds number, where inertial terms were neglected in the conservation equations. The results indicated that the flow of a Newtonian fluid over a finite-thickness viscoelastic gel in the limit of zero Reynolds number is unstable, and the mechanism of instability is the transfer of energy from mean flow to fluctuations due to the deformation work done by mean flow at the interface. It should be noted that this result is qualitatively different from that of

Yih (1967) and Hooper & Boyd (1983) for the interface between two fluids, and Kumaran & Srivatsan (1998) for the flow past a membrane, because inertial effects were neglected, and the growth rate is not proportional to the Reynolds number. Therefore, this is a zero Reynolds number instability where the flow becomes unstable even in the absence of inertia. The important difference between the studies of Kumaran (1995) and Kumaran & Srivatsan (1998) is that tangential motion of the interface was permitted in the former, whereas it was not permitted in the latter. Therefore, it is of interest to determine whether tangential motion of the membrane surface could result in an instability in the absence of inertia. This issue is of importance because most membranes in biological systems do have some tangential motion. In lipid bilayers, for example, the tangential motion would result in a variation in the areal density (number per unit area) of the lipid molecules in the membrane, and this would be resisted by an in-plane stress in the membrane. A similar motion could also take place in surfactant monolayers at surfaces. The effect of tangential motion on the stability of the flow past a membrane surface is examined here. It is shown that the flow could become linearly unstable when tangential motion is permitted in the absence of inertia, consistent with the results of Kumaran & Srivatsan (1998) which indicated that the flow is stable at zero Reynolds number in the absence of tangential motion.

The fluid flow past a membrane cannot strictly be treated as a flow in which the properties are invariant along the flow direction, because the tangential stress exerted by the fluid in the base state causes a variation in the tension of the membrane, and the membrane tension is a function of the tangential position. However, in the present analysis, we consider flows for which the fluid strain rate at the surface is small compared to the characteristic strain rate ($\Gamma^*/\eta^*/R^*$) for the system, where Γ^* is the surface tension, η^* is the fluid viscosity and R^* is the width of the fluid layer (the convention used here is that parameters with a superscript * are dimensional, while parameters without the superscript are non-dimensional). In this case, the variation of the membrane tension over distances comparable to the wavelength of the perturbations is small, and this variation can be neglected in comparison to the mean tension. More specifically, the rate of change of membrane tension in the tangential direction scales as $\eta^*\dot{\gamma}^*$, where $\dot{\gamma}^*$ is the strain rate in the fluid at the membrane surface. The analysis reveals that the most unstable modes have wavenumber $\alpha^* \ll R^{*-1}$, and the dimensional strain rate for these $\dot{\gamma}^* \propto R^*\alpha^{*2}\Gamma^*/\eta^*$. Since the wavelength of the most unstable modes scales as α^{*-1}, the variation of the membrane tension over length scales comparable to the wavelength of the most unstable mode is $R^*\alpha^*\Gamma^*$. This variation is small compared to the membrane tension for $\alpha^* \ll R^{*-1}$, and consequently the membrane tension is considered to be a constant over lengths comparable to the wavelength of the most unstable mode.

The condition that the flow-induced tension is neglected in comparison with the mean tension in the membrane results in a lower bound on the allowable wavenumbers. This can be seen as follows: the ratio of flow-induced tension and the imposed mean membrane tension is given by ($\eta^*\dot{\gamma}^*L^*/\Gamma^*$) which is equal to $\Lambda L^*/R^*$ where $\Lambda = \dot{\gamma}^*R^*\eta^*/\Gamma^*$ is the non-dimensional strain rate, and L^* is the lateral extent of the membrane. The condition for flow-induced tension to be less than the membrane tension is given by $R^*\alpha^* \gg \Lambda$, since the longest allowable wavelength has a wavenumber $\alpha^* \propto (1/L^*)$. This puts a lower bound on the allowable wavenumbers in the $\alpha^* \to 0$ limit. However, the analysis shows that the non-dimensional destabilizing velocity $\Lambda \sim (\alpha^{*2}R^{*2})$ in the $\alpha^* \to 0$ limit, and the condition that ($\alpha^*R^* \gg \Lambda$) is satisfied and the analysis is self-consistent in the low-wavenumber limit.

An examination of the numerical values of membrane and fluid parameters indicates that this parameter regime is encountered in biological systems. An additional parameter that enters into the analysis for compressible membranes is the modulus of elasticity along the surface, K^*. Using estimates $K^* = 10^{-1}\,\mathrm{N\,m^{-1}}$ and a tension of $\Gamma^* = 10^{-3}\,\mathrm{N\,m^{-1}}$ corresponding to a linear extension ratio of 1%, the non-dimensional modulus of elasticity is $K = K^*/\Gamma^* \sim 100$. The non-dimensional strain rate causing instability $\Lambda \sim \alpha^2(\Gamma^*/\eta^* R^*)$. For realistic values of $\eta^* = 10^{-4}\,\mathrm{kg\,m^{-1}\,s^{-1}}$, $\Gamma^* = 10^{-3}\,\mathrm{N\,m^{-1}}$ and $R^* = 10^{-4}\,\mathrm{m}$, and $\alpha \sim 10^{-2}$ corresponding to wavelengths of around 100 times the channel width, the calculations indicate that the dimensional strain rate for unstable fluctuations is $\dot{\gamma}^* \sim 10\,\mathrm{s^{-1}}$. This corresponds to a maximum velocity of $10^{-3}\,\mathrm{m\,s^{-1}}$ in a channel of width $10^{-4}\,\mathrm{m}$, and this is certainly in the range of velocities encountered in biological systems. The variation of flow-induced tension over a wavelength is $10^{-5}\,\mathrm{N\,m^{-1}}$ and can be neglected in comparison with the imposed mean tension.

The effect of nonlinear interactions on the growth of the perturbations is examined using a weakly nonlinear analysis. The nonlinear terms can saturate the exponential growth of disturbances in a linearly unstable system, leading to a supercritically stable state, thereby taking a linearly unstable system to a new stable state. This occurs in Rayleigh–Bérnard convection in a fluid heated from below, and in the Taylor–Couette instability in a rotating fluid. In other instances, a linearly stable system can become unstable to finite-amplitude disturbances due to nonlinear interactions. The bifurcation is then called subcritical and is known to occur in plane Poiseuille flow.

Two approaches have been used previously for the weakly nonlinear analysis of parallel flows. The weakly nonlinear theory of Stuart (1960) is applicable to finite but small disturbances, so that the nonlinearities can be treated perturbatively. This involves the assumption that only the fundamental mode exists in the system at the onset of instability and that higher harmonics of the fundamental mode are generated due to nonlinear interactions. The amplitudes of the higher harmonics are then expanded in an asymptotic series in the amplitude of the most unstable mode. The second approach involves the full numerical simulation of the time evolution of an arbitrary disturbance imposed on the laminar flow and so there is no restriction on amplitude of disturbances.

There has been some work on the nonlinear stability of flow past flexible surfaces (Pierce 1992; Rotenberry 1992; Rotenberry & Saffman 1990). All these studies have considered the Tollmien–Schlichting instability which exists in the flow past a rigid surface but which is modified due to the flexibility of the wall medium. Pierce (1992) derived the Ginzburg–Landau equation using the weakly nonlinear analysis for the case of plane Poiseuille flow in a channel with compliant walls of finite thickness. The result of Pierce (1992) indicated that the bifurcation is subcritical, but the Landau constant was found to vary significantly with the variation of wall parameters. Rotenberry & Saffman (1990) studied the weakly nonlinear stability of plane Poiseuille flow in a channel with compliant walls. The compliant walls were modelled as spring-backed plates, and only normal motion was permitted in the wall. They derived the appropriate Ginzburg–Landau equation using the method of Stewartson & Stuart (1971), and their results showed that the flow is subcritically unstable in the limit of rigid walls, which was in agreement with the previous studies on a rigid channel. However, when the wall is made compliant, there is a cross-over from subcritical instability to supercritical equilibration. Rotenberry (1992) studied the finite-amplitude stability of flow in a channel with compliant walls (modelled as spring-backed walls without tangential motion) by numerically calculating the travelling wave solutions that bifurcate along the neutral stability curve. The disturbance stream function was expanded in a Fourier series and only the first four Fourier modes were taken into account in the numerical calculation. Thomas (1992) studied the weakly nonlinear stability of Blasius flow past a spring-backed wall using a triple-deck asymptotic analysis in the limit of high *Re*. This study also considered the Tollmien–Schlichting modes and the results indicated the presence of supercritical equilibrium-amplitude states. All these studies have examined the limit of high Reynolds number, and in this limit nonlinearities are present in both the governing equations and boundary conditions. Recently, Shankar & Kumaran (2001) carried out the weakly nonlinear analysis of flow over gel at low Reynolds number. They found that the results were sensitive to the boundary conditions applied at the lower surface of the gel. For a 'grafted gel' where zero displacement conditions were applied, the bifurcation was subcritical. However for the case of 'adsorbed gel', where zero tangential stress conditions are applied at the lower surface, the system was supercritically stable for a large range of parameters η_r, the ratio of gel to fluid viscosity, and H, the gel thickness. The bifurcation changed from supercritical to subcritical as the wavenumber was increased. The results for the grafted gel were verified in the experiments of Kumaran & Muralikrishnan (2000).

The linear stability characteristics of the present system differ from those of Stuart (1960) and Shankar & Kumaran (2001), because for an imposed fluid strain rate Λ, perturbations with wavenumber less than a transition value $\alpha = C\sqrt{\Lambda}$ are unstable, where C is a constant. This is in contrast to earlier studies cited above where the most unstable mode has a finite wavenumber, and is similar to the stability of the interface between two fluids (Joseph & Renardy 1993). Though an unbounded system is unstable in the limit $\Lambda \to 0$, a bounded system of size L has a critical velocity for transition from stable to unstable modes when $\sqrt{\Lambda} > (2\pi/LC)$, and the wavenumber of the most unstable mode is $(2\pi/L)$. In this case, there is a discrete spectrum in the low-wavenumber limit, consisting of the fundamental mode with wavenumber $(2\pi/L)$ and its harmonics. In these cases, it is known that the stabilization of supercritical states could take one of two forms. In cases where the mode with the largest growth rate has a first harmonic which is stable, a time-dependent two-mode equilibrium is achieved where the mode with the largest growth rate is stabilized by its first harmonic. In other cases, the nonlinear interactions result in states where energy is continually exchanged between different two-mode states. The linear stability analysis indicates that the present instability falls into the former category, where the first harmonic of the linearly fastest growing mode is stable. Therefore, we carry out a weakly nonlinear analysis to determine whether the nonlinear interaction stabilizes this two-mode state.

The present work adopts the approach of Stuart (1960) and Shankar & Kumaran (2001) for the shear flow of two Newtonian fluids past a membrane. However, the approach used here is simpler for the following reason. In Stuart (1960), zero velocity boundary conditions are applied at the walls of the channel. If the equation for the linear growth rate of the perturbations is reduced to a linear equation of the form $\mathscr{L}\phi = 0$, where $\mathscr{L}$ is a linear operator and ϕ is a dynamical variable, it is necessary to determine the adjoint operator $\mathscr{L}^*$ and then apply the Fredholm alternative theorem. In the present case, the boundary conditions at the membrane surface are used to obtain the dispersion matrix, and the determinant of this matrix is set equal to zero to determine the linear growth rate. Therefore, it is sufficient to determine the adjoint of the dispersion matrix of boundary conditions in the weakly nonlinear analysis.

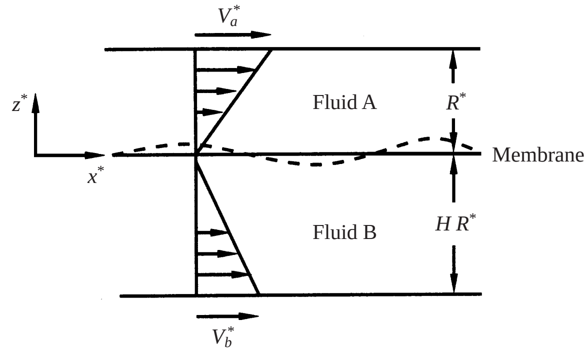

FIGURE 1. Schematic of the membrane configuration.

The problem formulation is described in § 2. Section 3 describes the weakly non-linear theory for the problem, and § 4 contains the principal results of the analysis. The details of the stability analysis are provided in Appendix A.

2. Problem formulation

The system, shown in figure 1, consists of a membrane of infinitesimal thickness and negligible inertia and surface tension Γ^* stretched along the interface $z^* = 0$ between two fluids A and B of thickness R^* and HR^* and viscosities η_a^* and η_b^* respectively. The surface bounding the fluid A at $z^* = R^*$ moves with a velocity V_a^*, while the surface bounding the fluid B at $z^* = -HR^*$ moves with a velocity V_b^* so that the undisturbed velocity profiles are given by $\bar{v}_x^{*a} = V_a^* z^*/R^*$ and $\bar{v}_x^{*b} = -V_b^* z^*/(R^* H)$. The fluid is described by the incompressible Stokes equations in the absence of inertia. Due to perturbations in the fluid stresses at the membrane surface, a material point on the surface undergoes a displacement from its steady-state position $(x^*, 0)$ to a new position $(x^* + \xi^*, \zeta^*)$, and the displacements in the x^*- and z^*-directions are denoted by u_x^* and u_z^* respectively. The relation between (ξ^*, ζ^*) and (u_x^*, u_z^*) is discussed a little later. It is important to note that (u_x^*, u_z^*) are the three-dimensional displacement fields, and not the displacement along the surface, which is denoted by u_s^*. The relation between the surface displacement u_s^* and the three-dimensional displacement fields (u_x^*, u_z^*) is also given a little later.

The dimensional surface stress tensor for the membrane $\sigma_{\alpha\beta}^*$ is related to the strain along the membrane surface $e_{\alpha\beta}^*$ by the constitutive relation (Harden & Pleiner 1994)

$$\sigma_{\alpha\beta}^* = (B^* + \eta_{bm}^* \partial_t)\delta_{\alpha\beta} e_{\gamma\gamma}^* + (G^* + \eta_{sm}^* \partial_t)(e_{\alpha\beta}^* - (1/2)\delta_{\alpha\beta} e_{\gamma\gamma}^*), \tag{2.1}$$

where membrane parameters B^* (bulk modulus) and G^* (shear modulus) are defined in units of (force/length), and η_{bm}^* (bulk viscosity) and η_{sm}^* (shear viscosity) are defined in units of (force/length × time). The tensor $e_{\alpha\beta}^*$ is the rate of deformation tensor in the plane of the membrane, and $\sigma_{\alpha\beta}^*$ is the two-dimensional stress tensor along the membrane surface which has units of (force/length). In the above equation, the dimensional rate of deformation tensor is expressed in terms of the displacement field along the membrane surface as

$$e_{\alpha\beta}^* = (1/2)(\partial_\alpha^* u_\beta^* + \partial_\beta^* u_\alpha^*). \tag{2.2}$$

For the present system, it can be proved that Squire's theorem holds so that two-dimensional perturbations are always more unstable than three-dimensional ones.

Therefore, it suffices to study the stability of the system to two-dimensional disturbances. In this case, the membrane tension can be written as

$$\sigma^*_{ss} = (K^* + \eta^*_m \partial^*_t) e^*_{ss}, \tag{2.3}$$

where the subscript s indicates the components along the membrane surface in the (x^*, z^*)-plane, and it is important to note that there is no summation in the above equation. Here, K^* is the effective elastic modulus of the membrane given by $K^* = B^* + (1/2)G^*$ and $\eta^*_m = \eta^*_{bm} + (1/2)\eta^*_{sm}$. The lengths are scaled by R^*, the velocities by (Γ^*/η^*_a), the time coordinate by $(R^*\eta^*_a/\Gamma^*)$, the fluid stresses and pressure by (Γ^*/R^*), and the membrane stresses by Γ^* to obtain a dimensionless equation for the stress along the membrane surface:

$$\sigma_{ss} = (K + \eta_m \partial_t) e_{ss} = (K + \eta_m \partial_t) \partial_s u_s, \tag{2.4}$$

where $\sigma_{ss} = \sigma^*_{ss}/\Gamma^*$, $K = K^*/\Gamma^*$ and $\eta_m = \eta^*_m / R^* \eta^*_a$. For two-dimensional perturbations with height variations in the x-direction, the displacement vector u_s along the membrane in the (x, z)-plane can be expressed in terms of the displacement fields (u_x, u_z) using simple geometric considerations,

$$u_s = u_x (1 + (\partial_x u_z)^2)^{1/2}, \tag{2.5}$$

and the gradient along the surface can be written as $\partial_s = \boldsymbol{t} \cdot \nabla$, where $\boldsymbol{t}$ is the tangent to the membrane surface and ∇ is the three-dimensional gradient operator.

The parameters $\Lambda_a = (V^*_a \eta^*_a / \Gamma^*)$ and $\Lambda_b = -(V^*_b \eta^*_a / \Gamma^* H)$ are defined so that the scaled mean velocities are

$$\bar{v}^l_x = \Lambda_l z, \tag{2.6}$$

where l is a for fluid A and b for fluid B. The equations governing the dynamics of fluids A and B are the usual incompressible Stokes equations in the zero Reynolds number limit, which are non-dimensionalized as above to provide

$$\partial_i v^l_i = 0, \tag{2.7}$$

$$-\partial_i p^l + \frac{\eta_l}{\eta_a} \partial^2_j v^l_i = 0, \tag{2.8}$$

where η_a and η_b are the non-dimensional viscosities of fluid A and B such that $\eta_a = 1$ and $\eta_b = \eta^*_b/\eta^*_a = \eta_r$. The boundary conditions at the interface are the continuity of normal and tangential velocities of the two fluids and the membrane, and the normal and tangential force balance conditions. The force balance requires that the difference between the tangential fluid stresses is balanced by the gradient in the tangential stress along the membrane, and the difference between the normal fluid stresses is balanced by the normal force due to membrane tension. Since the position of the interface has to be determined as a part of the solution, the boundary conditions at the perturbed interface are expanded about their values at the unperturbed interface $z = 0$. If F is a fluid parameter (fluid velocity, stress), and M is a parameter defined on the membrane surface (membrane displacement, stress), $F|_{x+\xi,\zeta}$ and $M_{x+\xi}$ at the perturbed interface are expanded in a Taylor series about their value at $(x, 0)$ (figure 2)

$$F|_{x+\xi,\zeta} = [F]_0 + [\partial_x F]_0 \xi + [\partial_z F]_0 \zeta + \tfrac{1}{2}[\partial^2_x F]_0 \xi^2 + \tfrac{1}{2}[\partial^2_z F]_0 \zeta^2 + [\partial_x \partial_z F] \zeta \xi + \cdots, \tag{2.9}$$

$$M|_{x+\xi} = [M]_0 + [\partial_x M]_0 \xi + \tfrac{1}{2}[\partial^2_x M]_0 \xi^2 + \cdots, \tag{2.10}$$

where $[\cdots]_0$ represent quantities evaluated at the unperturbed interface, and ξ and ζ

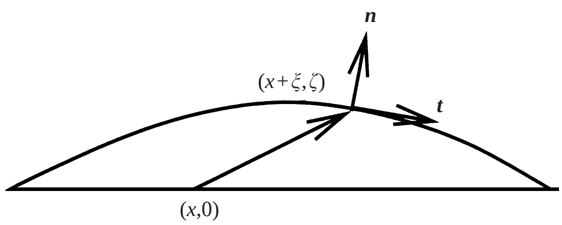

FIGURE 2. Configuration of the perturbed interface with the unit vectors.

are obtained as a part of the solution,

$$\xi \equiv u_x(x+\xi,t) = [u_x]_0 + [\partial_x u_x]_0 \xi + \tfrac{1}{2}[\partial_x^2 u_x]_0 \xi^2 + \cdots, \tag{2.11}$$

$$\zeta \equiv u_z(x+\xi,t) = [u_z]_0 + [\partial_x u_z]_0 \xi + \tfrac{1}{2}[\partial_x^2 u_z]_0 \xi^2 + \cdots. \tag{2.12}$$

From the above expressions, infinite series representations for ξ and ζ are obtained, and these are truncated at the required order in the weakly nonlinear theory. The velocity field in the membrane (v_i^m) is defined as the substantial derivative of the displacement field,

$$v_i^m = \partial_t u_i + v_j^m \partial_j u_i. \tag{2.13}$$

For the x- and z-components of the membrane velocity v_i^m, the above expressions become

$$v_x^m = \partial_t u_x + v_x^m \partial_x u_x, \tag{2.14}$$

$$v_z^m = \partial_t u_z + v_x^m \partial_x u_z. \tag{2.15}$$

The above equations are solved to determine v_x^m and v_z^m,

$$v_x^m = \frac{\partial_t u_x}{1-\partial_x u_x} = \partial_t u_x(1 + \partial_x u_x + (\partial_x u_x)^2 + O((\partial_x u_x)^3), \tag{2.16}$$

$$v_z^m = \partial_t u_z + \frac{\partial_t u_x}{1-\partial_x u_x}\partial_x u_z = \partial_t u_z + \partial_t u_x\, \partial_x u_z(1 + \partial_x u_x + O((\partial_x u_x)^2)). \tag{2.17}$$

The unit normal $\boldsymbol{n}$ and the unit tangent $\boldsymbol{t}$ to the perturbed interface are defined as (figure 2)

$$\boldsymbol{n} = \frac{-\boldsymbol{e}_x(\partial\zeta/\partial x) + \boldsymbol{e}_z}{\sqrt{1+(\partial\zeta/\partial x)^2}}, \quad \boldsymbol{t} = \frac{\boldsymbol{e}_x + \boldsymbol{e}_z(\partial\zeta/\partial x)}{\sqrt{1+(\partial\zeta/\partial x)^2}}. \tag{2.18}$$

The matching conditions for the velocity and stress at the perturbed interface $(x+\xi,\zeta)$ are

$$(t_i v_i^a)|_{x+\xi,\zeta} = (t_i v_i^b)|_{x+\xi,\zeta} = (t_i v_i^m)|_{x+\xi,\zeta}, \tag{2.19}$$

$$(n_i v_i^a)|_{x+\xi,\zeta} = (n_i v_i^b)|_{x+\xi,\zeta} = (n_i v_i^m)|_{x+\xi,\zeta}, \tag{2.20}$$

$$[n_i\tau_{ij}n_j]^a_{x+\xi,\zeta} - [n_i\tau_{ij}n_j]^b_{x+\xi,\zeta} = (\nabla_s \cdot \boldsymbol{n})_{x+\xi,\zeta}, \tag{2.21}$$

$$[t_i\tau_{ij}n_j]^a_{x+\xi,\zeta} + [t_i\tau_{ij}n_j]^b_{x+\xi,\zeta} + \partial_s^2 \sigma_{ss}|_{x+\xi,\zeta} = 0, \tag{2.22}$$

where the superscript m refers to variables defined on the membrane surface, ∇_s is the gradient along the membrane surface in the (x,z)-plane, and σ is the stress tensor along the membrane.

The terms in the boundary conditions (2.19), (2.20), (2.21), (2.22) are expanded in Taylor series in the parameters u_x and u_z. Due to the perturbation to the interface

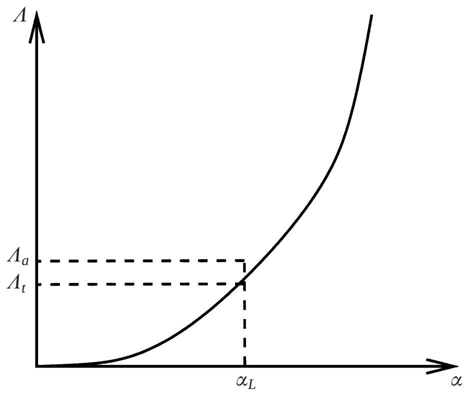

FIGURE 3. Schematic for the variation of amplitude with wavenumber for a constant top-plate velocity Λ_a.

position, the normal stress balance (2.21) contains the shear stress (τ_{xz}) of the mean flow, while the normal stress τ_{zz} appears in the tangential stress balance (2.22).

3. Theory

In this section, the weakly nonlinear theory is developed for flow of two fluids over a viscoelastic membrane in the zero Reynolds number limit. The governing equations (2.7) and (2.8) are linear, and nonlinearities arise due to the Taylor expansion of the boundary conditions about the unperturbed state, as well as due to the variation of the surface normal due to the perturbations. The perturbations to the velocity and stress fields are expressed using the function $E(x,t) = \exp[\mathrm{i}(\alpha_t x + \omega t)]$ where ω is the frequency of the perturbations. In the weakly nonlinear theory, an expansion is used in the harmonic series,

$$\phi(x,z,t) = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(A_1(\tau))^n[E^k\tilde{\phi}^{(k,n)}(z) + E^{-k}\tilde{\phi}^{(k,n)\dagger}(z)], \qquad (3.1)$$

where $\tilde{\phi}^{(0,0)}$ is the value of the variable in the mean flow, the superscript† denotes a complex conjugate, $A_1(\tau) = \epsilon A(\tau)$ is the amplitude of the wave which varies on the slow time scale τ (to be defined below), ϵ is a small parameter which gives the amplitude of perturbations, and $A(\tau)$ is an $O(1)$ quantity. It should be noted that $A_1(\tau)$ and $A(\tau)$ are real, since the temporal oscillations are included in $E(x,t)$.

In the vicinity of the transition point for the linear problem (Λ_t, α_t), the amplitude is governed by the Landau expansion,

$$A_1(\tau)^{-1}\,\mathrm{d}_t A_1(\tau) = s_r^{(0)} + A_1(\tau)^2 s_r^{(1)} + \cdots, \qquad (3.2)$$

where $s_r^{(0)}$ is the real part of the linear growth rate $s^{(0)}$, $s_r^{(1)}$ is the real part of the first Landau constant $s^{(1)}$, and Λ_t is the strain rate at transition. In the present analysis, we restrict attention to the case where $\Lambda_b = 0$ for the bottom fluid, and the transition strain rate Λ_t corresponds to the strain rate of the top fluid A at the onset of instability. For a system of lateral extent L^*, the lowest possible wavenumber is $\alpha_L = (2\pi R^*/L^*)$, and perturbations with this wavenumber become unstable when the dimensionless strain rate is $\Lambda_a = \Lambda_t$, as shown schematically in figure 3. When Λ_a is equal to Λ_t, the real part of the growth rate is identically zero.

When Λ_a is slightly larger than Λ_t ($\Lambda_a - \Lambda_t \ll \Lambda_a$), the real part can be written as $s_r = (\mathrm{d}s_r/\mathrm{d}\Lambda_a)|_{\Lambda_a=\Lambda_t}(\Lambda_a - \Lambda_t)$, where the velocity difference $\Lambda_a - \Lambda_t$ drives the linear instability. If $s_r^{(1)}$ is $O(1)$, then the second term in the right-hand side of (3.2) is $O(\epsilon^2)$, and a balance is achieved if $(\Lambda_a - \Lambda_t)\,\mathrm{d}s_r^{(0)}/\mathrm{d}\Lambda_a \sim \epsilon^2$, where ϵ is the small parameter which gives the amplitude of the perturbations. For definiteness, let $(\Lambda_a - \Lambda_t) = \Lambda_2\epsilon^2$, where Λ_2 is $O(1)$. This term is balanced by the term on the left-hand side of (3.2), and so we introduce the slow time scale in the time derivative as $\mathrm{d}_t = \epsilon^2\,\mathrm{d}_\tau$. Multiple time scales arise because there is a fast time scale (t) corresponding to the inverse of the frequency of oscillations, and a slow time scale (τ) over which the amplitude of the perturbations grows. At the neutral stability curve, the real part of the growth rate is zero, and very near the neutral stability curve it is expected that the time scale for the growth of perturbations is long compared to the period of oscillations. The amplitude is then assumed to vary over the slow time scale while the frequency is incorporated explicitly in the expansion with a faster time scale (3.1). Since $A_1(\tau)$ is independent of the fast time scale t, the above equation becomes

$$A^{-1}\,\mathrm{d}_\tau A = \Lambda_2\,\mathrm{d}s_r^{(0)}/\mathrm{d}\Lambda_a + s_r^{(1)}A^2. \tag{3.3}$$

This is the 'scaled' version of the Landau equation in the vicinity of the critical point of the linear neutral curve. Similarly, the frequency of oscillations ω is also expanded in the following series:

$$\omega = s_i^{(0)} + A(\tau)^2 s_i^{(1)} + \cdots, \tag{3.4}$$

where $s_i^{(0)}$ is the frequency of perturbations obtained from the linear theory and $s_i^{(1)}$ is the modification to the frequency of the perturbations due to self-interactions.

All the dynamical quantities are expanded in the amplitude and harmonic expansions as in equation (3.1). In the limit of zero inertia, the governing equations are linear and the equations for $\tilde{\phi}^{(k,n)}$ do not contain any inhomogeneous terms. However, as stated before, all the boundary conditions are expanded about the unperturbed interface and this results in inhomogeneous terms. The boundary conditions for the problem at order (k,n) contain inhomogeneous terms of order (k,m) where $m < n$, and the original nonlinear problem with an unknown interface is reduced to a hierarchy of linear (but inhomogeneous) problems, which are solved beginning from the linear $(1,1)$ problem. Although the expansion in equation (3.1) is general, the assumption of weakly nonlinear analysis ensures that only $(1,1), (0,2), (2,2)$ and $(1,3)$ problems need to be solved. The $(1,1)$ problem is the linear stability analysis, the $(0,2)$ problem is the correction to the mean flow due to the nonlinearities, the $(2,2)$ problem is the nonlinear correction to the first harmonic, while the $(1,3)$ problem is the nonlinear correction to the fundamental, at which order the Landau equation is recovered. The details of the solution procedure are provided in Appendix A. As discussed in the introduction, the solution procedure in the present case is easier than that of Stuart (1960) and Shankar & Kumaran (2001) because the governing equations are linear and the boundary conditions at the membrane surface are used to obtain the dispersion matrix, and the determinant of this matrix is set equal to zero to determine the linear growth rate. Therefore, it is sufficient to determine the adjoint of the dispersion matrix in the weakly nonlinear analysis, instead of determining the adjoint of the linear operator as in the case of Stuart (1960). The two methods were verified to give identical results, and the solution procedure using the method of Stuart (1960) is described in Appendix B. Here, we have followed the simpler procedure of finding the adjoint of the matrix of boundary conditions.

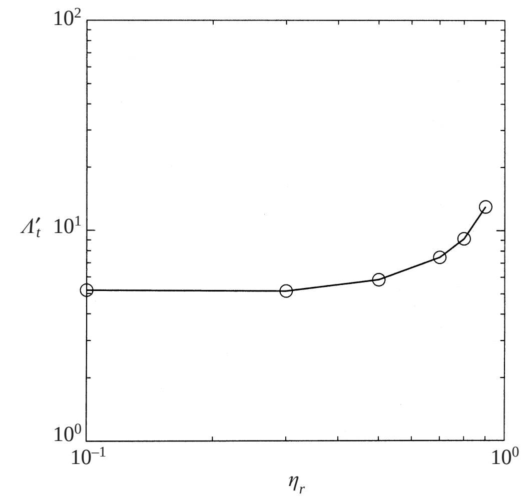

FIGURE 4. Effect of relative viscosity η_r on Λ_t' ($K = 102, \eta_m = 0, H = 1.0$).

4. Results

The linear stability analysis shows the presence of an instability even in the limit of zero Reynolds number, for certain membrane and fluid parameters, and the velocity for transition from stable to unstable modes Λ_t increases as $\Lambda_t'\alpha^2$ in the $\alpha \to 0$ limit, where $\Lambda_t'(K, H, \eta_r, \eta_m)$ is a coefficient which depends on the membrane and fluid properties. This indicates that small-wavenumber perturbations are unstable at any non-zero velocity in this parameter regime. Results are restricted to the case where there is a non-zero strain rate in the top fluid A, while the bottom fluid B is considered to be stationary. The expression for the real part of the growth rate near the neutral stability curve for the case of $H = 1$ in the $\alpha \to 0$ limit is

$$s_r = -\frac{\alpha^4}{12K(1+\eta_r)^3}\left[K(1+\eta_r)^2 + 3(-1 - 6\eta_r + 7\eta_r^2)\frac{\Lambda_a^2}{\alpha^4}\right], \tag{4.1}$$

and the transition velocity is given by

$$\Lambda_t = \alpha^2\sqrt{\frac{K\,(1+\eta_r)^2}{3(1+6\eta_r - 7\eta_r^2)}}. \tag{4.2}$$

Figures 4, 5 and 6 show the variation of the neutral stability curves with fluid and membrane parameters. Figure 4 indicates that perturbations are stabilized by an increase in the relative viscosity $\eta_r = \eta_b^*/\eta_a^*$. The transition velocity Λ_t' diverges as $\Lambda_t' \sim (1-\eta_r)^{-1/2}$ for $\eta_r \to 1$, and perturbations are stable for $\eta_r \geqslant 1.0$. The effect of the bottom fluid thickness H on the transition velocity is complicated. The expression for the transition velocity for $\eta_r = 1/2$ as a function of H in the $\alpha \to 0$ limit is

$$\Lambda_t' = \frac{(1+2H^3)\sqrt{K}}{\sqrt{3H\,(4H^5 + 4H^3 + 8H^2 - 3H - 4)}}. \tag{4.3}$$

The expression for Λ_t in the limit of $H \gg 1$ is given by $\sqrt{K}/(2\sqrt{3})(2 - 1/H^2)$, while the system is always stable when the bottom fluid thickness is less than $1/\sqrt{2}$. Figure 5 shows that the system becomes more stable as the height H of the bottom fluid is decreased and there is a critical value of H below which the system is always stable.

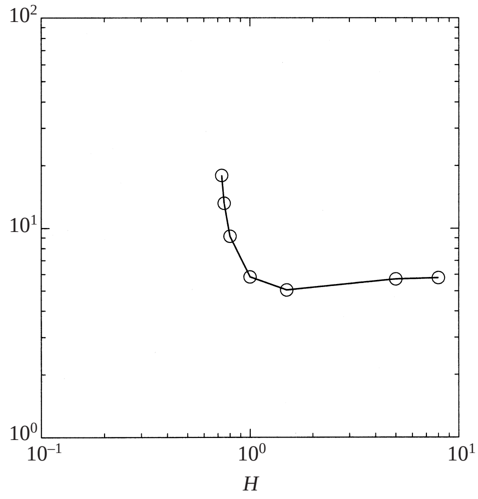

FIGURE 5. Effect of top fluid height H on Λ_t' ($\eta_r = 0.5, \eta_m = 0, K = 102.0$).

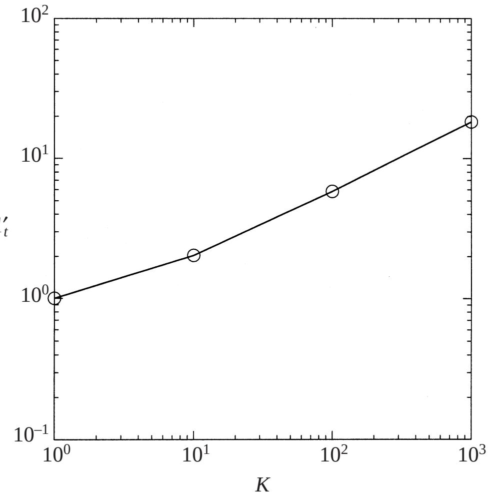

FIGURE 6. Effect of membrane elasticity of the membrane K on Λ_t' ($\eta_r = 0.5, \eta_m = 0, H = 1.0$).

The effect of the elastic modulus K, shown in figure 6, indicates that the system is stabilized by an increase in the elastic modulus of the membrane. The transition velocity Λ_t' diverges as $\sqrt{K}$, indicating that the flow past a membrane with no tangential motion is always stable. This is in agreement with the earlier results of Kumaran & Srivatsan (1998). The results also show that an increase in the scaled membrane viscosity η_m stabilizes perturbations, but the stabilization is not appreciable for $\eta_m < 10^2$.

The frequency of the neutrally stable mode is proportional to α^3 in the $\alpha \to 0$ limit, and the frequency is given by

$$\omega_t = -\frac{\alpha^3}{2}\sqrt{\frac{K}{3(1 + 6\eta_r - 7\eta_r^2)}} \tag{4.4}$$

for the case of $H = 1$ and $\eta_m = 0$. As indicated by equation (4.4), perturbations travel downstream for all the parameter values considered here.

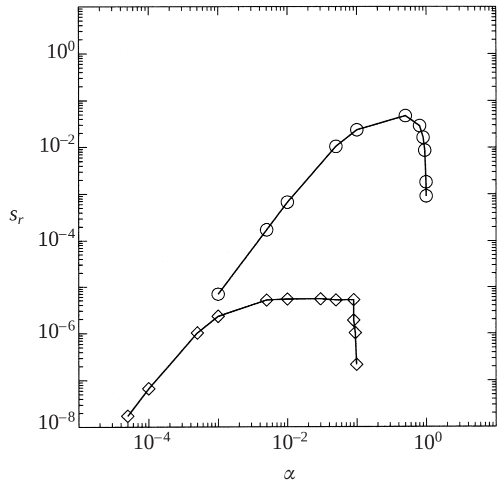

FIGURE 7. Variation of real part of growth rate s_r with α for different strain rate Λ_t (○, $\Lambda_t = 5.95$; ◇, $\Lambda_t = 0.058$, $K = 102$, $\eta_r = 0.5$, $\eta_m = 0$, $H = 1.0$).

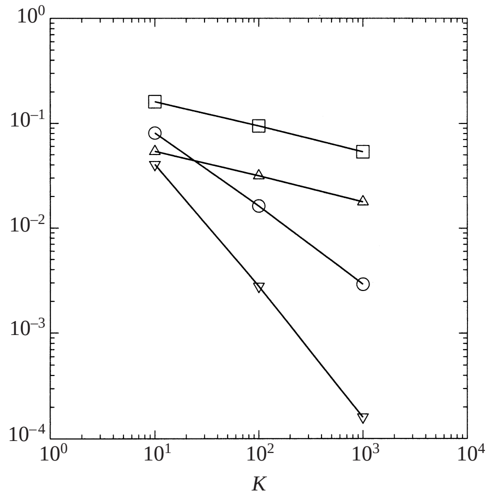

FIGURE 8. Variation of modified equilibrium amplitudes with membrane elasticity K (○, $\Psi = u_z/\sqrt{\Lambda - \Lambda_c}$; ▽, $\Psi = u_x/\sqrt{\Lambda - \Lambda_c}$; △, $\Psi = v_z/\alpha^3\sqrt{\Lambda - \Lambda_c}$; □, $\Psi = v_x/(\alpha^2\sqrt{\Lambda - \Lambda_c})$, $H = 1.0$, $\eta_r = 0.5$, $\eta_m = 0$).

Figure 7 shows the real part of the growth rate s_r vs. α for fixed values of fluid and membrane parameters. When the dimensionless wavenumber of the fastest growing mode is $O(1)$, the s_r vs. α plot shows a distinct maximum corresponding to the fastest growing mode. However, for low critical wavenumbers corresponding to low strain rates, the behaviour is unusual as there is no single fastest growing mode present in the system. Instead there is a band of wavenumbers near the critical wavenumber have nearly equal growth rates.

Though the low-wavenumber analysis predicts that there is an instability at zero wavenumber, in a real system the lowest permissible wavenumber is $(2\pi/L)$, where L is the system size in the lateral direction scaled by the channel thickness. The wavenumber spectrum in this case is discrete, consisting of the mode with wavenumber $(2\pi/L)$

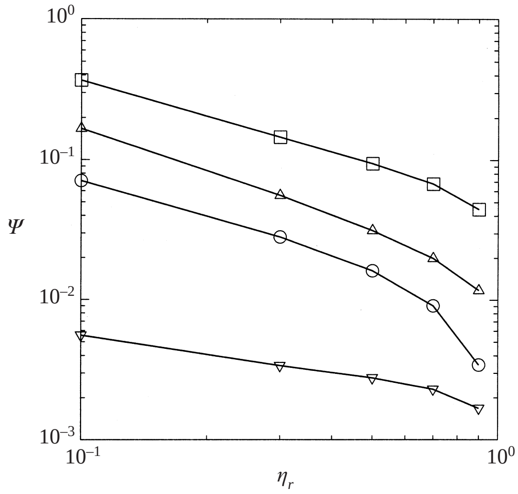

FIGURE 9. Variation of modified equilibrium amplitudes with relative fluid viscosity η_r, (symbols as figure 8, $H = 1.0$, $K = 100$, $\eta_m = 0$).

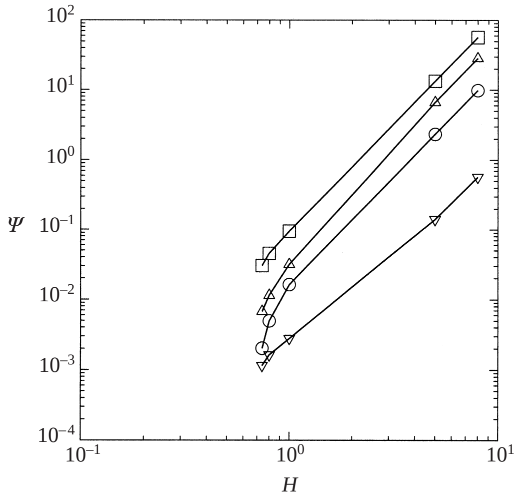

FIGURE 10. Variation of modified equilibrium amplitudes with height of bottom fluid H (symbols as figure 8, $K = 100$, $\eta_r = 0.5$, $\eta_m = 0$).

and its harmonics. The mode with wavenumber $(2\pi/L)$ becomes unstable for a finite velocity $\Lambda_a > \Lambda_t'(2\pi/L)^2$. In the regime $\Lambda_t'(2\pi/L)^2 < \Lambda_a < \Lambda_t'(4\pi/L)^2$, the mode with the lowest permissible wavenumber is unstable, but its first harmonic is stable. In this regime, a weakly nonlinear stability analysis is carried out to determine whether the nonlinear interactions stabilize the linearly unstable mode. The equilibrium amplitude obtained from equation (3.2) is given by

$$A_1^{eq} = \sqrt{\frac{(\mathrm{d}s_r^0/\mathrm{d}\Lambda_t)(\Lambda_a - \Lambda_t)}{s_r^1}}. \tag{4.5}$$

The analysis indicates that the Landau coefficient s_r^1 is always negative for the parameter values considered here, indicating that the linearly unstable mode is supercritically stable, and the equilibrium amplitude of the fluctuations scales as $\sqrt{(\Lambda_a - \Lambda_t)}$. Figures 8–10 show the variation of $u_z/\sqrt{\Lambda_a - \Lambda_t}$, $v_z/(\alpha^3\sqrt{\Lambda_a - \Lambda_t})$, $u_x/\sqrt{\Lambda_a - \Lambda_t}$, $v_x(\alpha^2\sqrt{\Lambda_a - \Lambda_t})$ with membrane parameters elastic modulus, relative viscosity, bottom fluid thickness and the damping coefficient. Figure 8 indicates that the equilibrium amplitude of the membrane displacements u_x and u_z and the membrane velocities v_x and v_z decrease with increase in elastic modulus. An increase in relative viscosity η_r and a decrease in the thickness of the bottom fluid H lead to an increase in the equilibrium amplitude (figures 9 and 10). The membrane damping does not appreciably affect the amplitude of perturbation. The frequency of oscillations given by the linear theory (equation (4.4)) has a small correction due to nonlinear interactions.

5. Conclusions

The stability of shear flow of two fluids past a viscoelastic membrane was studied using linear and weakly nonlinear analysis. The linear stability analysis of this problem shows that the flow is unstable to small perturbations for a wide range of fluid and membrane parameters in the limit of zero Reynolds number. The destabilizing velocity Λ_t increases proportional to α^2 in the limit of $\alpha \to 0$. Though the linear stability analysis predicts that there is an instability at zero wavenumber, in a real system the mode with the lowest permissible wavenumber $(2\pi/L)$ becomes unstable for a finite velocity $\Lambda_a > \Lambda_t'(2\pi/L)^2$. In the regime $\Lambda_t'(2\pi/L)^2 < \Lambda_a < \Lambda_t'(4\pi/L)^2$, the mode with the lowest permissible wavenumber is unstable, but its first harmonic is stable. A weakly nonlinear stability analysis was carried out in this regime.

The weakly nonlinear analysis for the system indicates that the system is supercritically stable in the $\alpha \to 0$ limit and the Landau constant for the system s_r^1 is negative. Thus the weakly nonlinear analysis indicates that the linearly unstable base state goes to a new oscillatory state. The equilibrium amplitude of the supercritically stable state is found to decrease with increase in the membrane elasticity and viscosity ratio while it increases with increase in H. Membrane damping however does not have a significant effect on the equilibrium amplitude.

Appendix A. Method of solution

A.1. *The $k = 1$, $n = 1$ problem*

The problem at order $k = 1$, $n = 1$ corresponds to the linear stability analysis of the flow of two fluids separated by a membrane. The governing equations in the two fluids at this order are

$$\partial_z \tilde{v}_z^{l(1,1)} + \mathrm{i}\alpha \tilde{v}_x^{l(1,1)} = 0, \tag{A 1}$$

$$-\mathrm{i}\alpha \tilde{p}_f^{l(1,1)} + \frac{\eta_l}{\eta_a}(\partial_z^2 - \alpha^2)\tilde{v}_x^{l(1,1)} = 0, \tag{A 2}$$

$$-\partial_z \tilde{p}_f^{l(1,1)} + \frac{\eta_l}{\eta_a}(\partial_z^2 - \alpha^2)\tilde{v}_z^{l(1,1)} = 0. \tag{A 3}$$

The boundary conditions at order $(1,1)$ are

$$\tilde{v}_z^{a(1,1)} = \tilde{v}_z^{b(1,1)}, \tag{A 4}$$

$$\tilde{v}_x^{a(1,1)} + \Lambda_a \tilde{u}_z^{(1,1)} = s^{(0)}\tilde{u}_x^{(1,1)}, \tag{A 5}$$

$$\tilde{v}_x^{b(1,1)} + \Lambda_b \tilde{u}_z^{(1,1)} = s^{(0)}\tilde{u}_x^{(1,1)}, \tag{A 6}$$

$$-\mathrm{i}\alpha\tilde{p}_f^{a(1,1)} + 2\partial_z\tilde{v}_z^{a(1,1)} = -\mathrm{i}\alpha\tilde{p}_f^{b(1,1)} + \frac{\eta_l}{\eta_a}2\partial_z\tilde{v}_z^{b(1,1)} + (2(\Lambda_a - \eta_r\Lambda_b)\mathrm{i}\alpha + \alpha^2)\tilde{u}_z^{(1,1)}, \quad \text{(A 7)}$$

$$\partial_z\tilde{v}_x^{a(1,1)} + \mathrm{i}\alpha\tilde{v}_z^{a(1,1)} = \frac{\eta_l}{\eta_a}(\partial_z\tilde{v}_x^{b(1,1)} + \mathrm{i}\alpha\tilde{v}_z^{b(1,1)}) + (K + 2\eta_m s^{(0)})\alpha^2\tilde{u}_x^{(1,1)}, \quad \text{(A 8)}$$

$$\tilde{v}_z^{a(1,1)} = s^{(0)}\tilde{u}_z^{(1,1)}, \quad \text{(A 9)}$$

where l is a for fluid A, b for fluid B and m for the membrane. Here, equation (A 4) gives the normal-velocity continuity condition while equations (A 5) and (A 6) are the tangential-velocity continuity conditions for the two fluids and the membrane. Equation (A 7) gives the normal-stress continuity where the last term contains two contributions to the normal stress. The first is due to mean shear stress coupling, which arises due to the variation in unit normal to the membrane, while the second is due to membrane tension. The tangential stress balance is given by equation (A 8). The eigenfunctions that are consistent with the boundary conditions at $z = 1$ and $z = -H$ are obtained analytically,

$$\tilde{v}_z^{a(1,1)} = A_1[\mathrm{e}^{\alpha z} - \mathrm{e}^{2\alpha-\alpha z}(1 - 2\alpha(1-z))] + A_2[\alpha\mathrm{e}^{(2\alpha-\alpha z)}(2\alpha(1-z) - z) + \alpha\mathrm{e}^{\alpha z}z], \quad \text{(A 10)}$$

$$\tilde{v}_z^{b(1,1)} = B_1[\mathrm{e}^{\alpha z} - \mathrm{e}^{-2\alpha H-\alpha z}(1+2\alpha(H-z))] + B_2[\alpha\mathrm{e}^{-2H\alpha-\alpha z}(2H\alpha(H-z)-z)+\alpha\mathrm{e}^{\alpha z}z]. \quad \text{(A 11)}$$

The eigenvalue of the linear problem $s^{(0)}$ and the constants in the eigenfunctions A_1, A_2, B_1, B_2 are determined from the boundary conditions at the interface $z = 0$. To determine all the four constants, an additional 'normalization' condition is required which we specify here as $|\tilde{u}_z^{a(1,1)}|_{z=0} = 1$. Thus, the (1, 1) eigenfunctions are obtained according to a specified normalization.

A.2. *The $k = 0$, $n = 2$ problem*

The $k = 0$, $n = 2$ problem represents the x-independent correction to the mean flow due to nonlinear interactions. The fluid continuity and the x and z momentum equations take the form

$$\partial_z\tilde{v}_z^{l(0,2)} = 0, \quad \text{(A 12)}$$

$$\partial_z^2\tilde{v}_x^{l(0,2)} = 0, \quad \text{(A 13)}$$

$$\partial_z\tilde{p}_f^{l(0,2)} + \frac{\eta_l}{\eta_a}\partial_z^2\tilde{v}_z^{l(0,2)} = 0. \quad \text{(A 14)}$$

Using the boundary condition $\tilde{v}_z^{a(0,2)} = 0$ at the top boundary $z = 1$, it can easily be seen that $\tilde{v}_z^{a(0,2)} = 0$ throughout the domain. Similarly, using the boundary condition at $z = -H$, it can easily be concluded that $\tilde{v}_z^{b(0,2)} = 0$. It is further convenient to set $\tilde{p}_f^{b(0,2)} = 0$ so that $\tilde{p}_f^{a(0,2)}$ is determined from the boundary conditions (A 14). The governing equations for $\tilde{v}_x^{a(0,2)}$ and $\tilde{v}_x^{b(0,2)}$ are linear (A 13), and the eigenfunctions can be written as

$$\tilde{v}_x^{a(0,2)}(z) = \mu_1 z + \tilde{v}_x^{a(0,2)}(0), \quad \text{(A 15)}$$

$$\tilde{v}_x^{b(0,2)}(z) = \mu_2 z + \tilde{v}_x^{b(0,2)}(0). \quad \text{(A 16)}$$

Here μ_1 and μ_2 are arbitrary constants, and $\tilde{v}_x^{a(0,2)}(0)$ and $\tilde{v}_x^{b(0,2)}(0)$ are the x velocities at the interface. Thus there are six unknowns to be determined: $\tilde{p}_f^{a(0,2)}$, $\tilde{u}_z^{(0,2)}$, μ_1, μ_2, $\tilde{v}_x^{a(0,2)}(0)$ and $\tilde{v}_x^{b(0,2)}(0)$.

The boundary conditions corresponding to the fluid normal-velocity continuity, tangential-velocity continuity of fluids A and B, normal-stress balance, tangential-stress balance and the fluid-membrane normal velocity continuity are given by equations (A 17), (A 18), (A 19), (A 20), (A 21) and (A 22) respectively. Note that for the

$(k=0, n=2)$ problem, all the perturbation variables are real, so that $\tilde{v}_x^{a(0,2)} = \tilde{v}_x^{*a(0,2)}$ and similarly for the other variables. The equations for the boundary conditions are

$$\tilde{v}_z^{a(0,2)} + \tilde{v}_z^{*a(0,2)} - \tilde{v}_z^{b(0,2)} - \tilde{v}_z^{*b(0,2)} + \mathrm{i}\alpha \tilde{u}_z^{*a(1,1)}\,\tilde{v}_x^{a(1,1)} - \mathrm{i}\alpha \tilde{u}_z^{(1,1)}\,\tilde{v}_x^{*a(1,1)} - \mathrm{i}\alpha \tilde{u}_z^{*(1,1)}\,\tilde{v}_x^{b(1,1)}$$
$$+\mathrm{i}\alpha \tilde{u}_z^{(1,1)}\,\tilde{v}_x^{*b(1,1)} + \mathrm{i}\alpha \tilde{u}_x^{*(1,1)}\,\tilde{v}_z^{a(1,1)} - \mathrm{i}\alpha \tilde{u}_x^{(1,1)}\,\tilde{v}_z^{*a(1,1)} - \mathrm{i}\alpha \tilde{u}_x^{*(1,1)}\,\tilde{v}_z^{b(1,1)} + \mathrm{i}\alpha \tilde{u}_x^{(1,1)}\,\tilde{v}_z^{*b(1,1)}$$
$$+\tilde{u}_z^{*(1,1)}\,\partial_z \tilde{v}_z^{a(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z \tilde{v}_z^{*a(1,1)} - \tilde{u}_z^{*(1,1)}\,\partial_z \tilde{v}_z^{b(1,1)} - \tilde{u}_z^{(1,1)}\,\partial_z \tilde{v}_z^{*b(1,1)} = 0, \qquad \text{(A 17)}$$

$$\Lambda_a\,\tilde{u}_z^{(0,2)} + \Lambda_a\,\tilde{u}_z^{*(0,2)} + \tilde{v}_x^{a(0,2)} + \tilde{v}_x^{*a(0,2)} - \mathrm{i}\alpha\Lambda_a\,\tilde{u}_x^{*(1,1)}\,\tilde{u}_z^{(1,1)} - \mathrm{i}\alpha\Lambda_a\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{*(1,1)}$$
$$-2\,\alpha\,(-\mathrm{i}s^0)\,\tilde{u}_z^{(1,1)}\,\tilde{u}_z^{*(1,1)} + \mathrm{i}\alpha \tilde{u}_x^{*(1,1)}\,\tilde{v}_x^{a(1,1)}\mathrm{i}\alpha \tilde{u}_x^{(1,1)}\,\tilde{v}_x^{*a(1,1)} - \mathrm{i}\alpha \tilde{u}_z^{*(1,1)}\,\tilde{v}_z^{a(1,1)}$$
$$+\mathrm{i}\alpha \tilde{u}_z^{(1,1)}\,\tilde{v}_z^{*a(1,1)} + \tilde{u}_z^{*(1,1)}\,\partial_z \tilde{v}_x^{a(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z \tilde{v}_x^{*a(1,1)} = 0, \qquad \text{(A 18)}$$

$$\Lambda_b\,\tilde{u}_z^{(0,2)} + \Lambda_b\,\tilde{u}_z^{*(0,2)} + \tilde{v}_x^{b(0,2)} + \tilde{v}_x^{*b(0,2)} - \mathrm{i}\alpha\Lambda_b\,\tilde{u}_x^{*(1,1)}\,\tilde{u}_z^{(1,1)}\mathrm{i}\alpha\Lambda_b\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{*(1,1)}$$
$$-2\,\alpha\,(-\mathrm{i}s^0)\,\tilde{u}_z^{(1,1)}\,\tilde{u}_z^{*(1,1)} + \mathrm{i}\alpha \tilde{u}_x^{*(1,1)}\,\tilde{v}_x^{b(1,1)} - \mathrm{i}\alpha \tilde{u}_x^{(1,1)}\,\tilde{v}_x^{*b(1,1)} - \mathrm{i}\alpha \tilde{u}_z^{*(1,1)}\,\tilde{v}_z^{b(1,1)}$$
$$+\mathrm{i}\alpha \tilde{u}_z^{(1,1)}\,\tilde{v}_z^{*b(1,1)} + \tilde{u}_z^{*(1,1)}\,\partial_z \tilde{v}_x^{b(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z \tilde{v}_x^{*b(1,1)} = 0, \qquad \text{(A 19)}$$

$$-\tilde{p}_f^{a(0,2)} - \tilde{p}_f^{*a(0,2)} + \tilde{p}_f^{b(0,2)} + \tilde{p}_f^{*b(0,2)} + 2\,\partial_z \tilde{v}_z^{a(0,2)} + 2\,\partial_z \tilde{v}_z^{a(0,2)} - 2\,\eta_r\,\partial_z \tilde{v}_z^{b(0,2)}$$
$$-2\,\eta_r\,\partial_z \tilde{v}_z^{b(0,2)} + \mathrm{i}\alpha \tilde{p}_f^{*a(1,1)}\,\tilde{u}_x^{(1,1)} - \mathrm{i}\alpha \tilde{p}_f^{*b(1,1)}\,\tilde{u}_x^{(1,1)} - \mathrm{i}\alpha \tilde{p}_f^{a(1,1)}\tilde{u}_x^{*(1,1)} + \mathrm{i}\alpha \tilde{p}_f^{b(1,1)}\tilde{u}_x^{*(1,1)}$$
$$-\mathrm{i}\alpha^3\,\tilde{u}_x^{*(1,1)}\,\tilde{u}_z^{(1,1)} + 2\alpha^2\,\Lambda_a\,\tilde{u}_x^{*(1,1)}\,\tilde{u}_z^{(1,1)} - 2\alpha^2\,\Lambda_b\,\eta_r\,\tilde{u}_x^{*(1,1)}\,\tilde{u}_z^{(1,1)} + \mathrm{i}\alpha^3 \tilde{u}_x^{(1,1)}\,\tilde{u}_z^{*(1,1)}$$
$$+2\alpha^2\,\Lambda_a\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{*(1,1)} - 2\alpha^2\,\Lambda_b\,\eta_r\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{*(1,1)} - 2\alpha^2 \tilde{u}_z^{*(1,1)}\,\tilde{v}_z^{a(1,1)} - 2\alpha^2 \tilde{u}_z^{(1,1)}\,\tilde{v}_z^{*a(1,1)}$$
$$+2\alpha^2\eta_r\,\tilde{u}_z^{*(1,1)}\,\tilde{v}_z^{b(1,1)} + 2\alpha^2\eta_r\,\tilde{u}_z^{(1,1)}\,\tilde{v}_z^{*b(1,1)} - \tilde{u}_z^{*(1,1)}\,\partial_z \tilde{p}_f^{a(1,1)} - \tilde{u}_z^{(1,1)}\,\partial_z \tilde{p}_f^{*a(1,1)}$$
$$+\tilde{u}_z^{*(1,1)}\,\partial_z \tilde{p}_f^{b(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z \tilde{p}_f^{*b(1,1)} + 2\mathrm{i}\alpha \tilde{u}_z^{*(1,1)}\,\partial_z \tilde{v}_x^{a(1,1)} - 2\mathrm{i}\alpha \tilde{u}_z^{(1,1)}\,\partial_z \tilde{v}_x^{*a(1,1)}$$
$$-2\mathrm{i}\alpha\eta_r \tilde{u}_z^{*(1,1)}\partial_z \tilde{v}_x^{b(1,1)} + 2\mathrm{i}\alpha\eta_r \tilde{u}_z^{(1,1)}\,\partial_z \tilde{v}_x^{*b(1,1)} + 2\mathrm{i}\alpha \tilde{u}_x^{*(1,1)}\,\partial_z \tilde{v}_z^{a(1,1)} - 2\mathrm{i}\alpha \tilde{u}_x^{(1,1)}\,\partial_z \tilde{v}_z^{a(1,1)}$$
$$-2\mathrm{i}\alpha\eta_r \tilde{u}_x^{*(1,1)}\,\partial_z \tilde{v}_z^{b(1,1)} + 2\mathrm{i}\alpha\eta_r \tilde{u}_x^{(1,1)}\,\partial_z \tilde{v}_z^{a(1,1)} + 2\,\tilde{u}_z^{*(1,1)}\,\partial_z^2 \tilde{v}_z^{a(1,1)} + 2\,\tilde{u}_z^{(1,1)}\,\partial_z^2 \tilde{v}_z^{*a(1,1)}$$
$$-2\,\eta_r\,\tilde{u}_z^{*(1,1)}\,\partial_z^2 \tilde{v}_z^{b(1,1)} - 2\,\eta_r\,\tilde{u}_z^{(1,1)}\,\partial_z^2 \tilde{v}_z^{*b(1,1)} = 0, \qquad \text{(A 20)}$$

$$2\mu_1 - 2\mu_3\,\eta_r + 4\alpha^3\,\eta_m\,(-\mathrm{i}s^0)\,\tilde{u}_x^{(1,1)}\,\tilde{u}_x^{*(1,1)} - 4\alpha^2\,\Lambda_a\,\tilde{u}_z^{(1,1)}\,\tilde{u}_z^{*(1,1)} + 4\alpha^2\,\Lambda_b\,\eta_r\,\tilde{u}_z^{(1,1)}\,\tilde{u}_z^{*(1,1)}$$
$$-2\alpha^2 \tilde{u}_z^{*(1,1)}\,\tilde{v}_x^{a(1,1)} - 2\alpha^2 \tilde{u}_z^{(1,1)}\,\tilde{v}_x^{*a(1,1)} + 2\alpha^2\eta_r\,\tilde{u}_z^{*(1,1)}\,\tilde{v}_x^{b(1,1)} + 2\alpha^2\eta_r\,\tilde{u}_z^{(1,1)}\,\tilde{v}_x^{*b(1,1)}$$
$$-\alpha^2\,\tilde{u}_x^{*(1,1)}\,\tilde{v}_z^{a(1,1)} - \alpha^2\,\tilde{u}_x^{(1,1)}\,\tilde{v}_z^{*a(1,1)} + \alpha^2\,\eta_r\,\tilde{u}_x^{*(1,1)}\,\tilde{v}_z^{b(1,1)} + \alpha^2\,\eta_r\,\tilde{u}_x^{(1,1)}\,\tilde{v}_z^{*b(1,1)}$$
$$+\mathrm{i}\alpha \tilde{u}_x^{*(1,1)}\,\partial_z \tilde{v}_x^{a(1,1)} - \mathrm{i}\alpha \tilde{u}_x^{(1,1)}\,\partial_z \tilde{v}_x^{*a(1,1)} - \mathrm{i}\alpha\,\eta_r\,\tilde{u}_x^{*(1,1)}\,\partial_z \tilde{v}_x^{b(1,1)} + \mathrm{i}\alpha\,\eta_r\,\tilde{u}_x^{(1,1)}\,\partial_z \tilde{v}_x^{*b(1,1)}$$
$$-\mathrm{i}\alpha \tilde{u}_z^{*(1,1)}\,\partial_z \tilde{v}_z^{a(1,1)} + \mathrm{i}\alpha \tilde{u}_z^{(1,1)}\,\partial_z \tilde{v}_z^{*a(1,1)} + \mathrm{i}\alpha\,\eta_r\,\tilde{u}_z^{*(1,1)}\,\partial_z \tilde{v}_z^{b(1,1)} - \mathrm{i}\alpha\,\eta_r\,\tilde{u}_z^{(1,1)}\,\partial_z \tilde{v}_z^{*b(1,1)}$$
$$+\tilde{u}_z^{*(1,1)}\,\partial_z^2 \tilde{v}_x^{a(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z^2 \tilde{v}_x^{*a(1,1)} - \eta_r\,\tilde{u}_z^{*(1,1)}\,\partial_z^2 \tilde{v}_x^{b(1,1)} - \eta_r\,\tilde{u}_z^{(1,1)}\,\partial_z^2 \tilde{v}_x^{*b(1,1)} = 0, \quad \text{(A 21)}$$

$$\tilde{v}_z^{a(0,2)} + \tilde{v}_z^{*a(0,2)} + \alpha\,(-\mathrm{i}s^0)\,\tilde{u}_x^{*(1,1)}\,\tilde{u}_z^{(1,1)} + \alpha\,(-\mathrm{i}s^0)\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{*(1,1)}$$

$$+\mathrm{i}\alpha\tilde{u}_z^{*(1,1)}\,\tilde{v}_x^{a(1,1)} - \mathrm{i}\alpha\tilde{u}_z^{(1,1)}\,\tilde{v}_x^{*a(1,1)} + \mathrm{i}\alpha\tilde{u}_x^{*(1,1)}\,\tilde{v}_z^{a(1,1)} - \mathrm{i}\alpha\tilde{u}_x^{(1,1)}\,\tilde{v}_z^{*a(1,1)}$$

$$+\tilde{u}_z^{*(1,1)}\,\partial_z\tilde{v}_z^{a(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_z^{*1(1,1)} = 0. \quad \text{(A 22)}$$

The fluid normal-velocity continuity condition at $z = 0$, and the membrane–fluid normal-velocity continuity condition, are identically satisfied by the solutions $\tilde{v}_z^{a(0,2)} = 0$ and $\tilde{v}_z^{b(0,2)} = 0$, as indicated by equations (A 17) to (A 22). The six unknowns can then be determined using the two tangential-velocity boundary conditions, the normal-stress boundary condition and the tangential-stress boundary condition, and two additional conditions are imposed:

$$\mu_1\,(1) + \tilde{v}_x^{a(0,2)}(0) = 0, \quad \text{(A 23)}$$

$$\mu_2\,(-H) + \tilde{v}_x^{b(0,2)}(0) = 0, \quad \text{(A 24)}$$

which correspond to the assumption that the top and bottom plate velocities are unchanged.

The solution of the $k = 0$, $n = 2$ problem suggests that the membrane can have a non-zero normal displacement ($\tilde{u}_z^{(0,2)}$) by generating a non-zero mean pressure in fluid A ($\tilde{p}_f^{a(0,2)}$). Thus the membrane moves to a new position in the z-direction, where the shear-stress balance is identically satisfied, and the tangential velocities are continuous. This is necessary, as it is postulated that the membrane does not develop any net mean tangential displacement in the x-direction and hence does not generate mean stresses. This solves the $k = 0$, $n = 2$ problem. The velocity $\tilde{v}_x^{(0,2)}$ represents the distortion of the laminar flow velocity profile due to the nonlinearities occurring at the boundary conditions, and this distortion is a linear function of z.

A.3. *The $k = 2$, $n = 2$ problem*

The $k = 2$, $n = 2$ problem represents the nonlinear correction to the first harmonic of the linearly unstable wavenumber α. The governing equations at order $k = 2$, $n = 2$ are

$$\partial_z\tilde{v}_z^{l(2,2)} + 2\mathrm{i}\alpha\tilde{v}_x^{l(2,2)} = 0, \quad \text{(A 25)}$$

$$-2\mathrm{i}\alpha\tilde{p}_f^{l(2,2)} + \frac{\eta_l}{\eta_a}(\partial_z^2 - 4\alpha^2)\tilde{v}_x^{l(2,2)} = 0, \quad \text{(A 26)}$$

$$-\partial_z\tilde{p}_f^{l(2,2)} + \frac{\eta_l}{\eta_a}(\partial_z^2 - 4\alpha^2)\tilde{v}_z^{l(2,2)} = 0. \quad \text{(A 27)}$$

These governing equations are supplemented by homogeneous boundary conditions at $z = 1$ and $z = -H$. At $z = 0$ there are six inhomogeneous boundary conditions. The boundary conditions corresponding to the fluid normal-velocity continuity, tangential-velocity continuity of fluids A and B, normal-stress balance, tangential-stress balance and the fluid–membrane normal-velocity continuity are given by equations (A 28), (A 29), (A 30), (A 31), (A 32) and (A 33) respectively,

$$\tilde{v}_z^{a(2,2)} - \tilde{v}_z^{b(2,2)} - \mathrm{i}\alpha\Lambda_a\,(\tilde{u}_z^{(1,1)})^2 + \mathrm{i}\alpha\Lambda_b\,(\tilde{u}_z^{(1,1)})^2 - \mathrm{i}\alpha\tilde{u}_z^{(1,1)}\,\tilde{v}_x^{a(1,1)} + \mathrm{i}\alpha\tilde{u}_z^{(1,1)}\,\tilde{v}_x^{b(1,1)}$$

$$+\mathrm{i}\alpha\tilde{u}_x^{(1,1)}\,\tilde{v}_z^{a(1,1)} - \mathrm{i}\alpha\tilde{u}_x^{(1,1)}\,\tilde{v}_z^{b(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_z^{a(1,1)} - \tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_z^{b(1,1)} = 0, \quad \text{(A 28)}$$

$$-2\mathrm{i}(-\mathrm{i}s^0)\,\tilde{u}_x^{(2,2)} + \Lambda_a\,\tilde{u}_z^{(2,2)} + \tilde{v}_x^{a(2,2)} + 2\alpha(-\mathrm{i}s^0)\,(\tilde{u}_x^{(1,1)})^2 + \mathrm{i}\alpha\Lambda_a\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{(1,1)}$$

$$+\alpha\,(-\mathrm{i}s^0)\,(\tilde{u}_z^{(1,1)})^2 + \mathrm{i}\alpha\tilde{u}_x^{(1,1)}\,\tilde{v}_x^{a(1,1)} + \mathrm{i}\alpha\tilde{u}_z^{(1,1)}\,\tilde{v}_z^{a(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_x^{a(1,1)} = 0, \quad \text{(A 29)}$$

$$-2\mathrm{i}(-\mathrm{i}s^0)\,\tilde{u}_x^{(2,2)} + \Lambda_b\,\tilde{u}_z^{(2,2)} + \tilde{v}_x^{b(2,2)} + 2\alpha(-\mathrm{i}s^0)\,(\tilde{u}_x^{(1,1)})^2 + \mathrm{i}\alpha\Lambda_b\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{(1,1)}$$
$$+\alpha\,(-\mathrm{i}s^0)\,(\tilde{u}_z^{(1,1)})^2 + \mathrm{i}\alpha\tilde{u}_x^{(1,1)}\,\tilde{v}_x^{b(1,1)} + \mathrm{i}\alpha\tilde{u}_z^{(1,1)}\,\tilde{v}_z^{b(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_x^{b(1,1)} = 0, \qquad \text{(A 30)}$$

$$-\tilde{p}_f^{a(2,2)} + \tilde{p}_f^{b(2,2)} - 4\alpha^2\,\tilde{u}_z^{(2,2)} - 4\mathrm{i}\alpha\Lambda_a\,\tilde{u}_z^{(2,2)} + 4\mathrm{i}\alpha\Lambda_b\,\eta_r\,\tilde{u}_z^{(2,2)} + 2\partial_z\tilde{v}_z^{a(2,2)}$$
$$-2\eta_r\,\partial_z\tilde{v}_z^{b(2,2)} - \mathrm{i}\alpha\tilde{p}_f^{a(1,1)}\,\tilde{u}_x^{(1,1)} + \mathrm{i}\alpha\tilde{p}_f^{b(1,1)}\,\tilde{u}_x^{(1,1)} - \mathrm{i}\alpha^3\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{(1,1)}$$
$$+2\alpha^2\,\Lambda_a\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{(1,1)} - 2\alpha^2\,\Lambda_b\,\eta_r\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{(1,1)} + 2\alpha^2\tilde{u}_z^{(1,1)}\,\tilde{v}_z^{a(1,1)} - 2\alpha^2\,\eta_r\,\tilde{u}_z^{(1,1)}\,\tilde{v}_z^{b(1,1)}$$
$$-\tilde{u}_z^{(1,1)}\,\partial_z\tilde{p}_f^{a(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z\tilde{p}_f^{b(1,1)} - 2\mathrm{i}\alpha\tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_x^{a(1,1)} + 2\mathrm{i}\alpha\eta_r\,\tilde{u}_z^{(1,1)}\,\tilde{v}_x^{b(1,1)}$$
$$+2\mathrm{i}\alpha\tilde{u}_x^{(1,1)}\,\partial_z\tilde{v}_z^{a(1,1)} - 2\mathrm{i}\alpha\eta_r\,\tilde{u}_x^{(1,1)}\,\partial_z\tilde{v}_z^{b(1,1)} + 2\tilde{u}_z^{(1,1)}\,\partial_z^2\tilde{v}_z^{a(1,1)}$$
$$-2\eta_r\,\tilde{u}_z^{(1,1)}\,\partial_z^2\tilde{v}_z^{b(1,1)} = 0, \qquad \text{(A 31)}$$

$$-4\alpha^2\,K\,\tilde{u}_x^{(2,2)} - 16\mathrm{i}\alpha^2\,\eta_m\,(-\mathrm{i}s^0)\,\tilde{u}_x^{(2,2)} + 2\mathrm{i}\alpha\tilde{v}_z^{a(2,2)} - 2\mathrm{i}\alpha\eta_r\,\tilde{v}_z^{b(2,2)} + \partial_z\tilde{v}_x^{a(2,2)}$$
$$-\eta_r\,\partial_z\tilde{v}_x^{b(2,2)} - \mathrm{i}\alpha^3\,K\,(\tilde{u}_x^{(1,1)})^2 + 2\alpha^3\,\eta_m\,(-\mathrm{i}s^0)\,(\tilde{u}_x^{(1,1)})^2 + 2\alpha^2\,\Lambda_a\,(\tilde{u}_z^{(1,1)})^2$$
$$-2\alpha^2\,\Lambda_b\,\eta_r\,(\tilde{u}_z^{(1,1)})^2 + 2\alpha^2\tilde{u}_z^{(1,1)}\,\tilde{v}_x^{a(1,1)} - 2\alpha^2\,\eta_r\,\tilde{u}_z^{(1,1)}\,\tilde{v}_x^{b(1,1)} - \alpha^2\,\tilde{u}_x^{(1,1)}\,\tilde{v}_z^{a(1,1)}$$
$$+\alpha^2\,\eta_r\,\tilde{u}_x^{(1,1)}\,\tilde{v}_z^{b(1,1)} + \mathrm{i}\alpha\tilde{u}_x^{(1,1)}\partial_z\tilde{v}_x^{a(1,1)} - \mathrm{i}\alpha\eta_r\,\tilde{u}_x^{(1,1)}\,\tilde{v}_x^{b(1,1)} + 3\mathrm{i}\alpha\tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_z^{a(1,1)}$$
$$-3\mathrm{i}\alpha\eta_r\,\tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_z^{b(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z^2\tilde{v}_x^{a(1,1)} - \eta_r\,\tilde{u}_z^{(1,1)}\,\partial_z^2\tilde{v}_x^{b(1,1)} = 0, \qquad \text{(A 32)}$$

$$\tilde{v}_z^{a(2,2)} - 2\mathrm{i}(-\mathrm{i}s^0)\,\tilde{u}_z^{(2,2)} + \alpha\,(-\mathrm{i}s^0)\,\tilde{u}_x^{(1,1)}\,\tilde{u}_z^{(1,1)} - \mathrm{i}\alpha\Lambda_a\,(\tilde{u}_z^{(1,1)})^2 - \mathrm{i}\alpha\tilde{u}_z^{(1,1)}\,\tilde{v}_x^{a(1,1)}$$
$$+\mathrm{i}\alpha\tilde{u}_x^{(1,1)}\,\tilde{v}_z^{a(1,1)} + \tilde{u}_z^{(1,1)}\,\partial_z\tilde{v}_z^{a(1,1)} = 0. \qquad \text{(A 33)}$$

The eigenfunctions for $\tilde{v}_z^{a(2,2)}$ and $\tilde{v}_z^{b(2,2)}$ can be obtained analytically, by the usual method of eliminating $\tilde{v}_x^{l(2,2)}$ and $\tilde{p}_f^{l(2,2)}$ to obtain a fourth-order ordinary differential equation for both the fluids. The top and bottom plate boundary conditions provide two constants for each fluid. The remaining four constants and the membrane displacement variables $\tilde{u}_x^{(2,2)}$ and $\tilde{u}_z^{(2,2)}$ can then be found using the six boundary conditions at the interface, as the linear system of equations at this order admits trivial solution to the homogeneous problem.

A.4. *The $k = 1$, $n = 3$ problem*

The variation of the amplitude $A(\tau)$ with the slow time scale appears as inhomogeneous terms in the boundary conditions at order $k = 1$, $n = 3$. The governing equations are identical to those for the linear problem $k = 1$, $n = 1$, since there are no nonlinear terms in the them.

The boundary conditions at order $(1, 3)$ are given by

$$\tilde{v}_z^{a(1,3)} - \tilde{v}_z^{b(1,3)} = g_1, \qquad \text{(A 34)}$$

$$\tilde{v}_x^{a(1,3)} + \Lambda_a\tilde{u}_z^{(1,3)} - s^{(0)}\tilde{u}_x^{(1,3)} = g_2, \qquad \text{(A 35)}$$

$$\tilde{v}_x^{b(1,3)} + \Lambda_b\tilde{u}_z^{(1,3)} - s^{(0)}\tilde{u}_x^{(1,3)} = g_3, \qquad \text{(A 36)}$$

$$(-\mathrm{i}\alpha\tilde{p}_f^{a(1,3)} + 2\partial_z\tilde{v}_z^{a(1,3)}) - \left(-\mathrm{i}\alpha\tilde{p}_f^{b(1,3)} + 2\frac{\eta_l}{\eta_a}\partial_z\tilde{v}_z^{b(1,3)}\right), \qquad \text{(A 37)}$$

$$-(2(\Lambda_a - \eta_r \Lambda_b)i\alpha + \alpha^2)\tilde{u}_x^{(1,3)} = g_4, \tag{A 38}$$

$$(\partial_z \tilde{v}_x^{a(1,3)} + \mathrm{i}\alpha\tilde{v}_z^{a(1,3)}) - \frac{\eta_l}{\eta_a}(\partial_z \tilde{v}_x^{b(1,3)} + \mathrm{i}\alpha\tilde{v}_z^{b(1,3)}), \tag{A 39}$$

$$-(K + 2\eta_m s^{(0)})\alpha^2 \tilde{u}_x^{(1,3)} = g_5, \tag{A 40}$$

$$\tilde{v}_z^{a(1,3)} - s^{(0)}\tilde{u}_z^{(1,3)} = g_6. \tag{A 41}$$

Here, g_1, g_2, g_3, g_4, g_5 and g_6 are the inhomogeneous terms in the boundary conditions which arise due to nonlinear interactions of the fundamental mode and its higher harmonics. The expressions for the inhomogeneities are very lengthy and are not provided here. In the above equations, g_2, g_3 and g_6 contain the slow time dependence of $A(\tau)$ on τ. Using the solutions $\tilde{v}_x^{l(1,3)}$ and $\tilde{v}_z^{l(1,3)}$ which satisfy the boundary conditions at $z = 1$ and $z = -H$, the interface boundary conditions (A 34) are written as $\mathscr{B}u = \mathscr{G}$. The left-hand side of the boundary conditions in the (1,3) problem ($\mathscr{B}u$) are identical to the (homogeneous) boundary conditions that appeared in the linear problem ($k = 1$, $n = 1$). The homogeneous problem $\mathscr{B}u = 0$ has non-trivial solutions for the $k = 1$, $n = 3$ problem. For the inhomogeneous $k = 1$, $n = 3$ problem to have non-trivial solution, Fredholm's alternative theorem states that the solution of the adjoint problem should be orthogonal to the inhomogeneous term $\mathscr{G}$. The solutions when substituted in the boundary conditions can then be written in the matrix form as

$$\mathbf{C} \cdot \boldsymbol{A} = \boldsymbol{G}, \tag{A 42}$$

where vector $\boldsymbol{A}$ is the matrix of constants in the boundary conditions for the (1,3) problem, $\mathbf{C} = (c_{ij})$ is the coefficient matrix and vector $\boldsymbol{G} = [g_1, g_2, g_3, g_4, g_5, g_6]$. The expression for the Landau coefficient is obtained using the solvability condition for the matrix equation. The adjoint problem for equation (A 42) is constructed by defining the inner product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ as

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \sum u_i^* v_i, \tag{A 43}$$

where u_i^* is the complex conjugate of u_i. Using the definition of adjoint we obtain

$$\mathbf{C}^+ \cdot \boldsymbol{A}^+ = 0, \tag{A 44}$$

where $\boldsymbol{A}^+ = [a1, a2, a3, a4, a5, a6]$ is the non-trivial adjoint solution for the homogeneous adjoint problem, and $\mathbf{C}^+ = (c_{ji}^*)$ is the adjoint of the matrix $\mathbf{C}$. The Landau equation is then obtained using the Fredholm solvability condition by setting the solution of the adjoint problem orthogonal to the inhomogeneities.

$$\boldsymbol{A}^+ \cdot \boldsymbol{G} = 0 \tag{A 45}$$

Appendix B. Method of Stuart (1960) to solve the $k = 1$, $n = 3$ problem

Consider the functions $\phi_1(z), \phi_2(z)$ and $\psi_1(z), \psi_2(z)$ such that they satisfy $\phi_1(1) = \partial_z\phi_1(1) = \phi_2(-H) = \partial_z\phi_2(-H) = 0$ and $\psi_1(1) = \partial_z\psi_1(1) = \psi_2(-H) = \partial_z\psi_2(-H) = 0$, and ϕ_1 and ψ_1 are defined in the domain $0 < z < 1$, and ϕ_2 and ψ_2 are defined in the domain $-H < z < 0$. The adjoint problem is constructed by multiplying $\mathscr{L}\phi$ by ψ and integrating from $-H$ to 1, where $\mathscr{L} \equiv (\partial_z^2 - \alpha^2)^2$. Integrating by parts yields

$$\int_0^1 \mathrm{d}z \psi_1 \mathscr{L}\phi_1 + \int_{-H}^0 \mathrm{d}z \psi_2 \mathscr{L}\phi_2 = \int_0^1 \phi_1 \mathscr{L}\psi_1 \,\mathrm{d}z + J_1|_{z=0} + \int_{-H}^0 \phi_2 \mathscr{L}\psi_2 \,\mathrm{d}z + J_2|_{z=0}. \tag{B 1}$$

Here J_1 and J_2 are the terms evaluated at the boundary $z=0$. They are given by

$$J_1 = -[\psi_1(\phi_1''' - 2\alpha^2\phi_1')]_{z=0} + [\psi_1'(\phi_1'' - 2\alpha^2\phi_1)]_{z=0} - [\phi_1'\psi_1'']_{z=0} + [\phi_1\psi_1''']_{z=0}, \quad (B\,2)$$

$$J_2 = [\psi_2(\phi_2''' - 2\alpha^2\phi_2')]_{z=0} - [\psi_2'(\phi_2'' - 2\alpha^2\phi_2)]_{z=0} + [\phi_2'\psi_2'']_{z=0} - [\phi_2\psi_2''']_{z=0}. \quad (B\,3)$$

Let $\phi_1 = \tilde{v}_z^{a(1,1)}$ and $\phi_2 = \tilde{v}_z^{b(1,1)}$. Then, the left-hand side of equation (B 1) vanishes because $\tilde{v}_z^{a(1,1)}$ and $\tilde{v}_z^{b(1,1)}$ identically satisfy the differential equations $\mathscr{L}\tilde{v}_z^{a(1,1)} = 0$ and $\mathscr{L}\tilde{v}_z^{(1,1)} = 0$. If we choose the adjoint function to satisfy $\mathscr{L}\psi_1 = 0$ in the range $0 < z < 1$ and $\mathscr{L}\psi_2 = 0$ in the range $-H < z < 0$, then the integrals on the right-hand side of (B 1) also vanish. Hence, the boundary conditions for the adjoint functions ψ_1 and ψ_2 at $z=0$ are determined by letting $J_1 + J_2 = 0$ so that we obtain the following definition for the adjoint operators:

$$\int_{-H}^{0} \psi_2\mathscr{L}\phi_2 + \int_0^1 \psi_1\mathscr{L}\phi_1 = \int_{-H}^{0} \phi_2\mathscr{L}\psi_2 + \int_0^1 \phi_1\mathscr{L}\psi_1. \quad (B\,4)$$

The condition $J_1|_{z=0} + J_2|_{z=0} = 0$ yields the following boundary conditions on the adjoint functions ψ_1 and ψ_2 at $z=0$. In $J_1|_{z=0} + J_2|_{z=0} = 0$, the values of ϕ_1 and its higher derivatives are expressed in terms of ϕ_2 and its higher derivatives using (A 34)–(A 41), where the inhomogeneities g are put to zero to obtain the homogeneous problem. Since ϕ_2 and its derivatives are non-zero at the interface, the boundary conditions for the adjoint eigenfunctions ψ_1 and ψ_2 are obtained by equating the coefficients of ϕ_2 and its derivatives, to zero, to give

$$\psi_2 = \eta_r\psi_1, \quad (B\,5)$$

$$\psi_1' = \frac{s^\dagger\psi_1(1-\eta_r)}{(K+2\eta_m s^\dagger)} + \frac{s^\dagger(\psi_1''-\psi_2'')}{\alpha^2(K+2\eta_m s^\dagger)}, \quad (B\,6)$$

$$\psi_2' = \eta_r\psi_1', \quad (B\,7)$$

$$\psi_2''' = \alpha^4\frac{\psi_1}{s^\dagger} + \frac{i\alpha^3\Lambda_a\psi_1}{s^\dagger} - \frac{\alpha^2 s^\dagger\psi_1(1-\eta_r)^2}{(K+2\eta_m s^\dagger)} - 2\alpha^2(\psi_1'-\psi_2')$$
$$-\frac{i\alpha\Lambda_a\psi_1''}{s^\dagger} - \frac{(\psi_1''(1-\eta_r) - \psi_2''(1-\eta_r))s^\dagger}{(K+2\eta_m s^\dagger)} + \psi_1''', \quad (B\,8)$$

where $s^\dagger$ is the eigenvalue of the adjoint problem and $\eta_r = \eta_b/\eta_a$. Using the above boundary conditions along with the differential equations $\mathscr{L}\psi_1 = 0$ and $\mathscr{L}\psi_2 = 0$, the eigenvalue $s^\dagger$ and the adjoint functions ψ_1 and ψ_2 are calculated.

The criterion for solvability (a variant of the Fredholm alternative theorem) for the inhomogeneous problem that occurs at $k=1$, $n=3$ is determined as follows. The governing equations for the $(1,3)$ problem are homogeneous, i.e. $\mathscr{L}\tilde{v}_z^{(1,3)} = 0$ and $\mathscr{L}\tilde{u}_z^{(1,3)} = 0$. Multiplying $\mathscr{L}\tilde{v}_z^{(1,3)}$ by the adjoint function ψ_1 and multiplying $\mathscr{L}\tilde{u}_z^{(1,3)}$ by the adjoint function ψ_2 and integrating over the appropriate domains, we obtain

$$\int_{-H}^{0} \psi_2\,\mathscr{L}\tilde{u}_z^{(1,3)}\,dz + \int_0^1 \psi_1\,\mathscr{L}\tilde{v}_z^{(1,3)}dz = \int_{-H}^{0} \tilde{u}_z^{(1,3)}\mathscr{L}\psi_2\,dz + \int_0^1 \tilde{v}_z^{(1,3)}\mathscr{L}\psi_1\,dz + \mathscr{J}_1 + \mathscr{J}_2, \quad (B\,9)$$

where $\mathscr{J}_1$ and $\mathscr{J}_2$ are the boundary terms evaluated at $z=0$ that contain (bilinear) products of $\tilde{v}_z^{(1,3)}$, ψ_1 and $\tilde{u}_z^{(1,3)}$ and ψ_2 and their derivatives. The left-hand side of (B 9) is identically zero since $\mathscr{L}\tilde{u}_z^{(1,3)} = \mathscr{L}\tilde{v}_z^{(1,3)} = 0$. The integrals in the right side are zero since the adjoint functions ψ_1 and ψ_2 are constructed to satisfy $\mathscr{L}\psi_1 = \mathscr{L}\psi_2 = 0$. Therefore, the necessary and sufficient condition for the existence of solutions at $k = 1$, $n = 3$ is given by

$$\mathscr{I}_1 + \mathscr{I}_2 = 0. \tag{B 10}$$

The terms $\mathscr{I}_1$ and $\mathscr{I}_2$ contain the inhomogeneities $g_1, g_2, g_3, g_4, g_5, g_6$ that occur in the boundary conditions of the $(1, 3)$ problem, and also the adjoint eigenfunctions ψ_1 and ψ_2 and their derivatives. These inhomogeneous terms, in turn, contain the slow variation of $A(\tau)$ with τ. Thus, the solvability condition that arises from the inhomogeneous nature of the boundary conditions determines the Landau equation at this order.

For the present purposes, it is useful to rewrite the inhomogeneous boundary conditions as follows:

$$\mathscr{F}_1 = \tilde{v}_z^{a(1,3)} - \tilde{v}_z^{b(1,3)}, \tag{B 11}$$

$$\mathscr{F}_2 = \partial_z \tilde{v}_z^{a(1,3)} - \frac{\mathrm{i}\alpha \Lambda_a \tilde{v}_z^{a(1,3)}}{s^{(0)}} - \frac{s^{(0)}}{\alpha^2(K + 2\eta_m s^{(0)})} \times [(\partial_z^2 \tilde{v}_z^{a(1,3)}) + \alpha^2 \tilde{v}_z^{a(1,3)}) - \eta_r(\partial_z^2 \tilde{v}_z^{b(1,3)} + \alpha^2 \tilde{v}_z^{b(1,3)})], \tag{B 12}$$

$$\mathscr{F}_3 = \partial_z \tilde{v}_z^{b(1,3)} - \frac{\mathrm{i}\alpha \Lambda_b \tilde{v}_z^{a(1,3)}}{s^{(0)}} - \frac{s^{(0)}}{\alpha^2(K + \eta_m s^{(0)})} \times [(\partial_z^2 \tilde{v}_z^{a(1,3)}) + \alpha^2 \tilde{v}_z^{a(1,3)}) - \eta_r(\partial_z^2 \tilde{v}_z^{b(1,3)} + \alpha^2 \tilde{v}_z^{b(1,3)})], \tag{B 13}$$

$$\mathscr{F}_4 = \partial_z^3 \tilde{v}_z^{a(1,3)} - \eta_r \partial_z^3 \tilde{v}_z^{b(1,3)} - 3\alpha^2(\partial_z \tilde{v}_z^{a(1,3)} - \eta_r \partial_z \tilde{v}_z^{b(1,3)}) + (2(\Lambda_a - \eta_r \Lambda_b)\mathrm{i}\alpha^3 + \alpha^4)\frac{\tilde{v}_z^{(1,3)}}{s^{(0)}}. \tag{B 14}$$

Here, $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$, $\mathscr{F}_4$ are related to the inhomogeneous functions (A 34)–(A 41) $g_1, g_2, g_3, g_4, g_5, g_6$, and the continuity and x momentum equations are used to eliminate the pressure terms. The extra two equations for the membrane displacement are replaced in terms of fluid velocities from the normal-velocity continuity and tangential-stress balance to give expressions for all the derivatives of $\tilde{v}_z^{(1,3)}$. These expressions are

$$\mathscr{F}_1 = g_1, \tag{B 15}$$

$$\mathscr{F}_2 = -\mathrm{i}\alpha\left(g_2 + \frac{\Lambda_a g_6}{s^{(0)}} - \frac{s^{(0)} g_5}{\alpha^2(K + 2\eta_m s^{(0)})}\right), \tag{B 16}$$

$$\mathscr{F}_3 = -\mathrm{i}\alpha\left(g_3 + \frac{\Lambda_b g_6}{s^{(0)}} - \frac{s^{(0)} g_5}{\alpha^2(K + 2\eta_m s^{(0)})}\right), \tag{B 17}$$

$$\mathscr{F}_4 = -\alpha^2\left(g_4 - \alpha^2 \frac{g_6}{s^{(0)}}\right). \tag{B 18}$$

Using the above for the inhomogeneous boundary conditions, the solvability condition for the existence of solutions, i.e. $\mathscr{I}_1 + \mathscr{I}_2 = 0$ for the case of $\Lambda_b = 0$ becomes, at $z = 0$,

$$\left[-(\alpha^2 F_2) - F_4 + \alpha^2 F_3 \eta_r + F_1\left(\frac{\alpha^4}{s^{(0)}} + \frac{\mathrm{i}\alpha^3 \Lambda}{s^{(0)}} - \frac{3\alpha^2 s^{(0)}}{\theta} + \frac{3\alpha^2 \eta_r s^{(0)}}{\theta}\right)\right]\psi_1(0) + \left[-F_2 - \frac{\mathrm{i}\alpha F_1 \Lambda}{s^{(0)}} - \frac{3F_1 s^{(0)}}{\theta}\right]\psi_1''(0) + \left[F_3 + \frac{3F_1 s^{(0)}}{\theta}\right]\psi_2''(0) + F_1 \psi_1^{(3)}(0) = 0. \tag{B 19}$$

In the above equation, the adjoint functions ψ_1 and ψ_2 are known from the solution of the adjoint problem, and $\theta = K + 2\eta_m s^{(0)}$. On substituting the adjoint functions ψ_1 and ψ_2, and taking the real part of the resulting equation, we obtain the Landau equation,

$$A(\tau)^{-1}\partial_\tau A(\tau) = \Lambda_2 \frac{\mathrm{d}s_r^{(0)}}{\mathrm{d}\Lambda_a} + s_r^{(1)} A(\tau)^2. \tag{B 20}$$

The imaginary part of the solvability condition gives the frequency of the waves,

$$\omega = s_i^{(0)} + \Lambda_2 \frac{\mathrm{d}s_i^{(0)}}{\mathrm{d}\Lambda_a} + s_i^{(1)} A(\tau)^2. \tag{B 21}$$

Here $s_r^{(1)}$ is the real part of the first Landau constant, which determines whether the instability is subcritical or supercritical and $s_i^{(1)}$ is the correction to the frequency of the basic wave due to nonlinear self-interactions. If $s_r^{(1)}$ is positive the instability is subcritical, while if $s_r^{(1)}$ is negative the instability is supercritical.

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